We consider the real case and the Lyapunov equation which is slightly different from the one considered in the book.

$$AW + WA^T = H$$

Suppose $H \geq 0$ and $W > 0$. Let λ be a (real/complex) eigenvalue of A and $x^*A = \lambda x^*$. According to Lemma 2.4.6, we have

$$\boldsymbol{Re(\lambda) = \frac{1}{2}\frac{x^*Hx}{x^*Wx}}$$

Therefore, $Re(\lambda) \geq 0$ and

$$\boldsymbol{Re(\lambda) = 0 \Leftrightarrow \begin{cases} x^*A = \lambda x^* \\ x^*H = 0 \end{cases}}$$

Suppose $H = BB^T$. Then $x^*H = 0 \Leftrightarrow x^*B = 0$. Therefore,

$$\boldsymbol{Re(\lambda) = 0 \Leftrightarrow \begin{cases} x^*A = \lambda x^* \\ x^*B = 0 \end{cases} \Leftrightarrow x^*[A - \lambda I, B] = 0}$$

The last equation actually is the PBH test for controllability!!! So (A, B) is controllable if and only if A is positive stable!

The negative stable case can be proved similarly.