We just consider the real case where A is real.

$$AW + WA^T = H$$

Now prove if $H \geq 0$ and $W > 0$, then A is positive semi-stable; i.e., $Re(\lambda) \geq 0$. The key is to analyze the matrix product AW!!!

Not complete proof: (for real eigenvalues and eigenvectors)

Let $\lambda \in \sigma(A)$ and $x^T A = \lambda x^T$ (for the above equation we need the left eigenvector, for the equation in the book, we need the right eigenvector). Now consider $x^T AWx$:

- $x^T AWx = \lambda x^T Wx$
- $x^T AWx = x^T \left(\frac{AW+WA^T}{2} + \frac{AW-WA^T}{2}\right) x = \frac{1}{2} x^T Hx$

Therefore,

$$\lambda x^T Wx = \frac{1}{2} x^T Hx \implies \lambda = \frac{1}{2} \frac{x^T Hx}{x^T Wx}$$

As a result, if W>0 and $H \geq 0$, then $\lambda \geq 0$; if $H > 0$, then $\lambda > 0$; if $H < 0$, then $\lambda < 0$.

Complete Proof: (for general complex eigenvalues and eigenvectors)

The above proof is not 100% correct because it is applicable merely to the case where λ and x are real. Even if A is real, its eigenvalues and eigenvectors can still be complex! If λ is complex (real), then x is complex (real)!

Let $\lambda \in \sigma(A)$ and $x^* A = \lambda x^*$. Now consider $x^* AWx$

- $x^* AWx = \lambda x^* Wx$
- $x^* AWx = x^* \left(\frac{AW+WA^T}{2} + \frac{AW-WA^T}{2}\right) x = \frac{1}{2}(x^* Hx + x^* Sx)$

Therefore,

$$\lambda x^* Wx = \frac{1}{2}(x^* Hx + x^* Sx)$$

It must be noted that $x^* Sx$ is not zero in the complex case. In fact, we have

- $x^* Sx$ is pure **imaginary** because $(x^* Sx)^* = -x^* Sx$
- $x^* Hx$ is pure **real** because $(x^* Hx)^* = x^* Hx$

Therefore, the above equation can be split to

$$2Re(\lambda) x^* Wx + 2Im(\lambda) x^* Wx = x^* Hx + x^* Sx \implies \begin{cases} 2Re(\lambda) x^* Wx = x^* Hx \\ 2Im(\lambda) x^* Wx = x^* Sx \end{cases}$$

As a result,

$$\boldsymbol{Re(\lambda) = \frac{1}{2} \frac{x^* Hx}{x^* Wx}}$$