How to Interpret Dilation and Cactus

Reference:

- Lin, 1976, Structural controllability
- YY Liu, 2015, Control principles of complex networks, Section II.C

Necessary and sufficient condition for structural controllability (SC)

1. The graph does not contain nonaccessible nodes or dilations
2. The graph is spanned by some cacti (cactus)

The notion of dilation is more difficult to understand. Roughly speaking, dilations are subgraphs in which a small subset of nodes attempts to rule a larger subset of nodes. It must be noted that T(S) can contains nodes that are contained in S. Otherwise, it is confusing for some networks.

We give examples here to demonstrate how to interpret the above two conditions.

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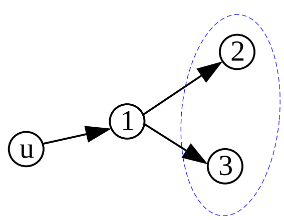
- Example: what is dilation

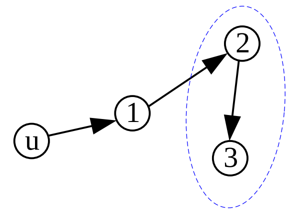


This network is not SC because it contains a dilation.

a. If $S = \{2,3\}$, then $T(S) = \{1\}$.
b. Mathematically, in the $[A, B]$ pair, there exists a tall nonzero sub-matrix.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Confusing example:



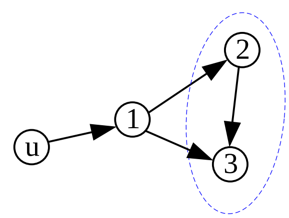
a. **Wrong conclusion**: if $S = \{2,3\}$, then $T(S) = \{1\}$, it has a dilation and hence is not SC.
b. **Correct conclusion**: if $\mathrm{S} = \{2,3\}$, then $\mathrm{T(S)} = \{1,2\}$. It does not contain a dilation. In fact, it is a stem.
c. Based on

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

It can be easily seen that it is controllable.

This example tells us that you clearly know what $T(S)$ is.

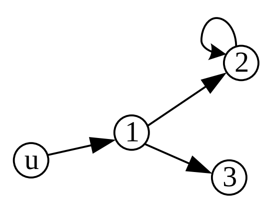
- Example:



a. Similar to the above example, here note that If $S = \{2,3\}$, then $T(S) = \{1,2\}$. So it does not contain a dilation. It is SC because it is spanned by a stem.
b.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Example:

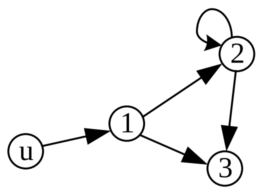


Adding self-loop can make the non-SC network SC.

a. With the selfloop, it does not have a dilation any more. That is because if $S = \{2,3\}$, then $T(S) = \{1,2\}$ instead of $T(S) = \{1\}$ because 2 points to itself. In fact, the network is a cactus: 1,3 form a stem and 1,2 form a bud.
b. Mathematically,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Confusing example:



If we look at

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

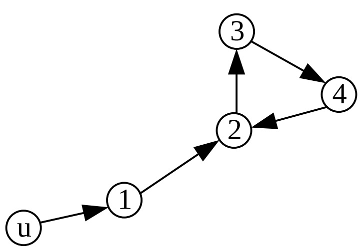
it is easy to draw the wrong conclusion that it is not SC. But in fact it is SC!!!

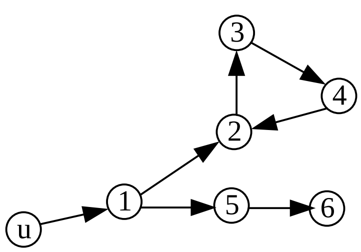
For some weights, it is not controllable. But we can make it controllable by changing the weight! It is called structurally controllable but not **strongly structurally controllable** (see reference 2). Besides, the network is spanned by a cactus.

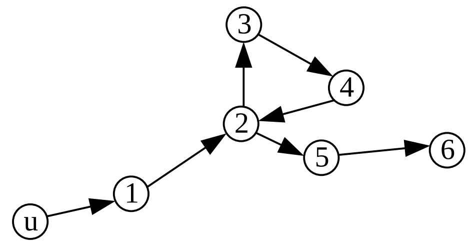
- Example: self-loop

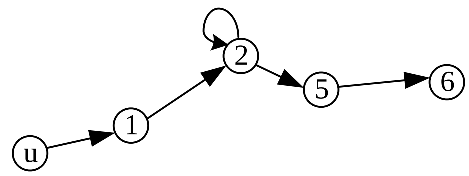
Adding self-loops will never make the system not SC. But it can make it not strongly structurally controllable. See the example above.

- Example: cactus









One simple conclusion is that ***tree graphs are not structurally controllable***.