Matrix Exponential of Jordan Block

If

$$J = \begin{bmatrix} \lambda & 1 & \dots & 0 \\ & \lambda & \ddots & \vdots \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \in \mathbb{R}^{k\times k}$$

then

$$e^{Jt} = e^{\lambda t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{k-1}}{(k-1)!} \\ & 1 & t & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ 0 & & & & 1 \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \dots & \frac{t^{k-1}}{(k-1)!}e^{\lambda t} \\ & e^{\lambda t} & te^{\lambda t} & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!}e^{\lambda t} \\ & & & \ddots & te^{\lambda t} \\ 0 & & & & e^{\lambda t} \end{bmatrix}$$

A. *Proof:*

Note

$$J = \lambda I_k + S$$

where

$$S = \begin{bmatrix} 0 & 1 & \dots & 0 \\ & 0 & \ddots & \vdots \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} \in \mathbb{R}^{k\times k}$$

is a *Shift matrix*, which is a special type of *nilpotent matrix*. Then we have

$$e^{Jt} = e^{\lambda t I_k + St} = e^{\lambda t I_k} e^{St} = e^{\lambda t} e^{St}$$

So the key is to compute e^{St}!!

The matrix S has nice properties such as $S^k = 0$ (the properties are given at the end of the article). Then we have

$$\begin{aligned} e^{St} &= I_k + St + \frac{t^2}{2!}S^2 + \dots + \frac{t^{k-1}}{(k-1)!}S^{k-1} + 0 + 0 + \dots \\ &= \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{k-1}}{(k-1)!} \\ & 1 & t & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ 0 & & & & 1 \end{bmatrix} \end{aligned}$$

Multiplying $e^{\lambda t}$ gives the result!

B. *Properties of the Shift Matrix* S

It can be seen that

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_{11} & a_{12} & a_{13} \\ 0 & a_{21} & a_{22} & a_{23} \\ 0 & a_{31} & a_{32} & a_{33} \\ 0 & a_{41} & a_{42} & a_{43} \end{bmatrix}$$

(How to remember: when S is on the left, it operates on the rows and when it is on the right, it operates one the columns.) Therefore, it can be seen that

$$S^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad S^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad S^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, we know

$$S^k = 0$$

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