Matrix Condition Number and Column Vector Angle

Consider the matrix $V = [v_1, \dots, v_n] \in \mathbb{R}^{n\times n}$ where $\|v_i\| = 1$. Suppose the vectors v_i are contained in a cone with the open angle as $\theta \in [0, \pi]$. As a result, the maximum angle between any two vectors is θ.

Main result:

$$\kappa(V) = \frac{\sigma_{\max}(V)}{\sigma_{\min}(V)} \geq \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

Remark:

- We can always flip the vectors in V to make that all the vectors are in a cone. The above inequality is valid for all matrix and conditions. Note the cone can have the open angle as π.
- The lower bound will be more sharp when the open angle is small. That is because the inequality is obtained from to special vectors. When the open angle is small, the two vectors will be close to the eigenvectors of the minimum and maximum singular eigenvalues.

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$$\sigma_{\min}(V) = \sqrt{\lambda_{\min}(VV^T)} \leq \sqrt{x^T VV^T x} = \|V^T x\|, \quad \forall x \in \mathbb{R}^n, \|x\| = 1$$

$$\sigma_{\max}(V) = \sqrt{\lambda_{\max}(VV^T)} \geq \sqrt{x^T VV^T x} = \|V^T x\|, \quad \forall x \in \mathbb{R}^n, \|x\| = 1$$

$$\|V^T x\| = \sqrt{\sum_{i=1}^{n} (v_i^T x)^2}$$

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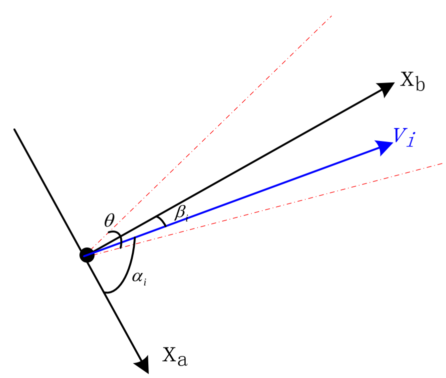

Now consider two special vectors x_a and x_b as shown in the figure.

- Note $v_i^T x_a = \cos\alpha_i$. Since

$$\frac{\pi}{2} - \frac{\theta}{2} \leq \alpha_i \leq \frac{\pi}{2}$$

  we have

$$0 \leq \cos\alpha_i \leq \cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \sin\frac{\theta}{2}$$

  As a result,

$$0 \leq \|V^T x_a\| = \sqrt{\sum_{i=1}^{n} (v_i^T x_a)^2} \leq \sqrt{n}\sin\frac{\theta}{2}$$

- Note $v_i^T x_b = \cos\beta_i$. Since

$$0 \leq \beta_i \leq \frac{\theta}{2}$$

we have

$$\cos\frac{\theta}{2} \leq \cos\beta_i \leq 1$$

As a result,

$$\sqrt{n}\cos\frac{\theta}{2} \leq \|V^T x_b\| = \sqrt{\sum_{i=1}^{n}(v_i^T x_b)^2} \leq \sqrt{n}$$

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According to $\|V^T x_a\|$ and $\|V^T x_b\|$, we have

$$\sigma_{\max}(V) \geq \sqrt{n}\cos\frac{\theta}{2}$$

$$\sigma_{\min}(V) \leq \sqrt{n}\sin\frac{\theta}{2} \text{ or } \sqrt{n}$$

Hence

$$\kappa(V) = \frac{\sigma_{\max}(V)}{\sigma_{\min}(V)} \geq \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}$$